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Oscillatory properties for semilinear degenerate hyperbolic equations of second order

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ABSTRACT

We consider three kind of oscillatory properties of the solutions to semilinear degenerate hyperbolic equations. Several sufficient conditions for the oscillation or non-oscillation are presented. In particular, they give us the positivity of the solutions for semilinear hyperbolic equations degenerating at initial point in one space dimension. Moreover we establish a few oscillatory conditions for the solutions of the mixed problem reduced to in one space dimension.

1. Introduction

In this article we are interested in some kind of oscillating properties of the solution to the Cauchy problem for the semilinear degenerate hyperbolic equations of second order

$$P_\pm(t,x,\partial_t,\partial_x)u = \partial_t^2 u - A(t,x,\partial_x)u \pm F(u) = 0,$$

whose terms are given by

$$A(t,x,\partial_x) = \sum_{j,k=1}^{n} \partial_{x_j}\bigl(a_{jk}(t,x)\partial_{x_k}\bigr), \qquad F(u) = b(t,x)u + f(u),$$

where $a_{jk}, b \in \mathcal{B}^\infty$, $f \in C^\infty$ are real-valued, $a_{jk} = a_{kj}$, $b \geqslant 0$ and $f(0) = 0$. $\mathcal{B}^\infty$ is the set of all infinitely differentiable functions whose derivatives are bounded. Moreover we always impose the following weak hyperbolicity conditions:

$$A_0(t,x,\xi) = \sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j\xi_k \geqslant 0$$

and a fortiori there exists a positive constant a_0 such that

$$\partial_t A_0(t,x,\xi) \geqslant -a_0 A_0(t,x,\xi) \tag{1.1}$$

for every $(t,x,\xi) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$.

Let us consider the Cauchy problem for $P_\pm$ with initial time $t_0 \geqslant 0$

$$\begin{cases} P_\pm u(t,x) = 0, & (t,x) \in [t_0,T) \times \mathbb{R}^n, \\ \partial_t^j u(t_0,x) = u_j(x), & x \in \mathbb{R}^n \ (j = 0,1), \end{cases} \tag*{$P(\pm)$}$$

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where $T \leqslant \infty$. We will devote to the study of the real-valued solution u to the problem $P(\pm)$ from now on. For this purpose, first recall the local solvability of $P(\pm)$ in Sobolev spaces. Let T^* be the life-span of the strong solution $u(t,x)$, that is,

$$T^* = \sup\left\{ T \in (t_0, \infty] \;\middle|\; \text{There is a unique solution } u(t,x) \in \bigcap_{j=0}^{1} C^j\big([t_0,T], H^{s-j}(\mathbb{R}^n)\big) \text{ of } P(\pm) \text{ with initial data } u_0, u_1 \in H_0^s(\mathbb{R}^n) \right\},$$

where $s \geqslant [n/2]+1$ (see, e.g., [4,7]). Throughout this article, denote $[t_0, T^*)$ by I.

Now, let us define a few oscillatory properties of $u(t,x)$ in the sense below:

(i) Globally Oscillatory Property (GOP): For any $t_0 \geqslant 0$ both

$$\mu\big(\big\{(t,x) \in I \times \mathbb{R}^n \;\big|\; u(t,x) > 0\big\}\big) > 0$$

and

$$\mu\big(\big\{(t,x) \in I \times \mathbb{R}^n \;\big|\; u(t,x) < 0\big\}\big) > 0$$

hold, where μ is the Lebesgue measure in $[0,\infty) \times \mathbb{R}^n$.

(ii) Pointwise Oscillatory Property (POP): For every fixed $x_0 \in \mathbb{R}^n$ there is a sequence $\{t_k\} \subset I$ made of distinct points which satisfies

$$u(t_k, x_0)u(t_{k+1}, x_0) < 0.$$

(iii) Non-Oscillatory Property (NOP): There exists some $T_0 \in I$ such that $u(t,x)$ does not change its sign for all $(t,x) \in [T_0, T^*) \times \mathbb{R}^n$.

There are some results on (GOP), for example, see [2,8,9,13,19]. But one knows few results on (POP) and (NOP) in comparison with (GOP). To our knowledge, (POP) are treated in Theorem 2 of [9] for linear P_- with time-independent coefficients in general space dimensions $n \geqslant 1$, a few fine results on (POP) for semilinear wave equations are investigated in [1] plus for $P(-)$ with time-dependent coefficients in [14] and [15] when $n = 1$. (NOP) is also discussed in [16] for $n \leqslant 3$ only because of the lack of the positivity of fundamental solutions to wave equations in high space dimensions $n \geqslant 4$. Recently, the monograph [18] on oscillation theory for various partial differential equations has published, which presents us an excellent survey on that subject. All the results mentioned above are related with strictly hyperbolic operators $P_\pm$ enjoying

$$A_0(t,x,\xi) \geqslant \delta|\xi|^2 \tag{1.2}$$

for some positive constant δ. Our main concern here is the case allowing $\delta = 0$ in (1.2).

First of all, we note our first result on (GOP). For this aim we need a few conditions on the coefficients of the operator P_+.

Assumptions 1.1.

(i) There is a constant b_0 such that $b(t,x) \geqslant b_0 > 0$ for all $(t,x) \in I \times \mathbb{R}^n$.
(ii) $f(s)s \geqslant 0$ for all $s \in \mathbb{R}$.

Theorem 1.1. *Under the Assumptions 1.1, if $T^* > \frac{\pi}{\sqrt{b_0}} + t_0$ and if there is a subinterval J of I with the length $|J|$ greater than $\frac{\pi}{\sqrt{b_0}}$ which the solution u to $P(+)$ is nontrivial in $J \times \mathbb{R}^n$, then (GOP) holds for the solution u.*

Remark 1.1. (i) Let us examine the case $\inf b(t,x) = 0$, for instance, $b,\ f \equiv 0$, $a_{11}(t) = t^{2\ell}$ $(\ell = 1,2,3,\dots)$ in one space dimension $n = 1$. Then, due to Corollary 3.3 in [17] we can represent the classical solution $u(t,x) \in C^2([0,\infty) \times \mathbb{R})$ of $P(+)$ with initial data $u_0 \equiv 0$, $u_1 \in C^2(\mathbb{R})$ by

$$u(t,x) = tc_\ell \int_0^1 \big\{u_1\big(x - \phi(t)s\big) + u_1\big(x + \phi(t)s\big)\big\}\big(1 - s^2\big)^{-\gamma}\, ds,$$

where c_ℓ is a positive constant only dependent on ℓ, $\phi(t) = \frac{t^{\ell+1}}{\ell+1}$ and $\gamma = \frac{\ell}{2(\ell+1)}$. In particular, if $u_1(x \pm \phi(t)s) > 0$ for $s \in [0,1]$, then $u(t,x) > 0$. This means that (GOP) to $P(+)$ is generally invalid for the case $\inf b(t,x) = 0$.

(ii) The optimality of the condition $|J| > \frac{\pi}{\sqrt{b_0+\delta}}$ in Theorem 1.1 is shown in Remark 1.4(a) of [2] (see also [8]) for strongly elliptic operators A like (1.2).

(iii) In general, $T^* < \infty$ since in one space dimension it is known that $P(+)$ even for some linear operator P_+ consisting of nonnegative only time-dependent coefficient $a_{11}(t) \in C^\infty$ and b, $f \equiv 0$ with suitable data in $C_0^\infty(\mathbb{R})$ is not locally solvable at the zeros of $a_{11}(t)$. This phenomenon is related to the oscillatory rate of $a_{11}(t)$ near its zeros. So under some restrictions on the vanishing orders of $a_{11}(t)$ and the growth of the nonlinear term f together with its derivatives, very few results on the time global solvability ($T^* = \infty$) for $P(+)$ are merely proved in low space dimensions (see [5,6] and the references therein). We shall not touch their hard problems at all.

Next we give our second result on (NOP) in one space dimension $n = 1$ only in the spirit of "comparison principle".

Assumptions 1.2.

(i) There are some point $T \in (t_0, T^*)$ and nonnegative function $a \in C^\infty([t_0, T])$ such that

$$A(t, x, \partial_x) = a(t)\partial_x^2$$

in $[t_0, T] \times \mathbb{R}^1$.

(ii) $a(t) > 0$ and $a'(t) \geqslant 0$ in $(t_0, T]$.

(iii) $F(s) \geqslant 0$ for $s \geqslant 0$.

Example 1.1. Let ℓ be a natural number. For instance, $a(t) = (t - t_0)^{2\ell}$ fulfills (i) and (ii) of Assumptions 1.2.

Theorem 1.2. *Let be $n = 1$. If Assumptions 1.2 are satisfied, then $u(t, x) > 0$ on $[t_0, T] \times \mathbb{R}^1$ for the solution u to $P(-)$ with positive initial data $u_0(x)$, $u_1(x)$.*

Further we shall state a result on (POP) for radially symmetric solutions in a ball B_ℓ to the equations $P_+u = 0$ in three space dimensions $n = 3$ only. In our strategy (POP) essentially needs some homogeneous boundary conditions. So, for simplicity, let us consider the mixed problem for P_+ with homogeneous Dirichlet boundary condition

$$\begin{cases} P_+u(t,x) = 0, & (t,x) \in [t_0, T] \times B_\ell \setminus \{0\}, \\ \partial_t^j u(t_0, x) = u_j(x), & x \in \overline{B_\ell}\ (j = 0, 1), \\ u(t,x) = 0, & \text{on } [t_0, T] \times |x| = 0, \ell, \end{cases} \qquad DP(+)$$

where $B_\ell = \{x \in \mathbb{R}^3 \mid |x| < \ell\}$. We have to restrict the coefficients of the operator $A(t, x, \partial_x)$ in order to reduce to the wave equation as well.

Assumptions 1.3.

(i) $a_{jk}(t, x)$ is only dependent on t, so $A(t, x, \partial_x) = a(t)\Delta_x$ with some nonnegative function $a(t) \in C^\infty([t_0, T])$ satisfying the conditions below:

(ii) $a(T) = 0$ and $a(t) > 0$ in $[t_0, T)$.

(iii) $b(t, x)$ is nonnegative and also only time-dependent, so say $b(t, x) = b(t)$.

(iv) $a''(t) \leqslant 4a(t)b(t) + \frac{5a'(t)^2}{4a(t)}$ holds in $[t_0, T)$. For example, $a''(t) \leqslant 0$ in $[t_0, T)$.

(v) $f(s)$ is odd, that is, $f(-s) = -f(s)$ in $\mathbb{R}$.

(vi) $f(s)$ is nondecreasing in $\mathbb{R}$.

Example 1.2.

(i) $a(t) = (T - t)^m$ satisfies (ii) and (iv) of Assumptions 1.3 when $m > 0$.

(ii) $a(t) = \exp(-(T - t)^{-\gamma})$ also does them if $\gamma > 0$ and if $T \leqslant (\frac{\gamma}{4\gamma+4})^{1/\gamma}$.

Theorem 1.3. *Assume that $T > 2\ell + t_0$ and that $v(t, r) = u(t, x)$ $(r = |x|)$ is a nontrivial radially symmetric solution to the mixed problem DP(+). Let $0 < |x_0| < \ell$. If Assumptions 1.3 are fulfilled, then for any $\hat{t} \in (t_0 + \ell, T - \ell)$ there exist two points $t_1, t_2 \in [\hat{t} - \ell, \hat{t} + \ell]$ such that $v(t_1, |x_0|)v(t_2, |x_0|) < 0$.*

Remark 1.2. (i) We merely know few conditions on f to be $T^* = \infty$ for $P(+)$. For instance, $T^* = \infty$ is possible for $a(t) = (T - t)^m$ and $f(u) = |u|^{p-1}u$ when $n = 3$ and $1 \leqslant p < \frac{3m+10}{3m+2}$ (see, in detail, (1.14) in [6]).

(ii) The conditions (ii), (iv) of Assumptions 1.3 are relaxed to the following. If the set $Z(a)$ of the zeros of $a(t)$ is discrete and if (ii), (iv) are fulfilled in some interval $[t_1, t_2)$ with $t_2 - t_1 > 2\ell$ of $[t_0, T] \setminus Z(a)$, then the conclusion of Theorem 1.3 also remains valid for any $\hat{t} \in (t_1 + \ell, t_2 - \ell)$.

In turn, let us show a result on (POP) for the mixed problem to the equations $P_+u = 0$ in one space dimensions $n = 1$ with homogeneous Neumann boundary condition

$$\begin{cases} P_+u(t,x) = 0, & (t,x) \in [t_0,\infty) \times (0,\ell), \\ \partial_t^j u(t_0,x) = u_j(x), & \text{in } [0,\ell]\ (j=0,1), \\ u_x(t,0) = u_x(t,\ell) = 0, & \text{on } [t_0,\infty). \end{cases} \qquad NP(+)$$

For this purpose, we must impose rather strong conditions on the coefficients of $A(t,x,\partial_x)$ and $f(u)$ to guarantee the global solvability for $NP(+)$ (see Theorem 1.3 in [6]). Some reduction to the wave equation is also used here.

Assumptions 1.4.

(i) $a_{jk}(t,x)$ is only dependent on t, so $A(t,x,\partial_x) = a(t)\partial_x^2$ with some nonnegative real-analytic function $a(t)$ defined on $[t_0,\infty)$ fulfilling the conditions below:

(ii) The set $Z(a)$ of the zeros of $a(t)$ is finite. Then we may assume that the order at each zero is finite as well, so is

$$m = \max\{k \in \mathbb{Z}_+ \mid a^{(k)}(t) = 0,\ t \in Z(a)\}.$$

(iii) $b(t,x)$ is also only time-dependent, so say $b(t,x) = b(t)$.

(iv) (Opial-type criterion, see (3.9) in [12].) Let $c(t) = \frac{b(t)}{a(t)} - \frac{a_{tt}(t)}{4a(t)^2} + \frac{5a_t(t)^2}{16a(t)^3}$ and $d(t) = \int_t^\infty c(s)\,ds$ for $t > T_0 = \max Z(a)$. Then the inequality

$$\int_t^\infty d^+(s)^2\,ds > \frac{1}{4}d(t) \tag{1.3}$$

holds and $d^+(t) \not\equiv 0$ for all $t > T_0$ large enough, where $d^+(t) = \max\{d(t),0\}$. Note that (1.3) is automatically valid at each t when $d(t) \leqslant 0$.

(v) $f(s)$ is odd, that is, $f(-s) = -f(s)$ in $\mathbb{R}$.

(vi) $f(s)$ is nondecreasing in $\mathbb{R}$.

(vii) The inequalities

$$\left|f^{(j)}(s)\right| \leqslant C_1\left(1+|s|\right)^{p-j}$$

and

$$\int_0^s f(\sigma)\,d\sigma \geqslant C_2|s|^{p+1}$$

are established for $j = 0,1$, some $C_1, C_2 > 0$ and $p \in [1, 1+4/m)$.

Remark 1.3. (i) The analyticity of $a(t)$ is used in place of (1.1) for the local solvability. Indeed, the condition (1.1) does not hold for $A(t,x,\partial_x)$ satisfying Assumptions 1.4. But the local solvability for $P(\pm)$ also remains valid because of (i) of Assumptions 1.4 (see [3]).

(ii) Of Assumptions 1.4 is equivalent to the condition $\sup Z(a) < \infty$ unless the real-analytic function $a(t)$ identically vanishes.

(iii) If $c(t) \geqslant 0$ for all t sufficiently large, then Hille's condition (see [12])

$$\liminf_{t\to\infty} t\,d(t) > \frac{1}{4}$$

implies (1.3).

Example 1.3. Let $b(t) \equiv 0$ and $a(t) = \frac{t^\alpha}{1+t^{\alpha+\beta}}$, where $\alpha > 0$ and $1 < \beta < 4$. Then, since

$$c(t) = \frac{\beta(\beta-4)t^{\alpha+2\beta-2}}{16(1+t^{\alpha+\beta})} + o\left(t^{\beta-2}\right) = \frac{\beta(\beta-4)}{16}t^{\beta-2} + o\left(t^{\beta-2}\right)$$

as $t \to \infty$, we see that $d(t) = \infty$ for t large enough and $\lim_{t\to\infty} t\,d(t) = \infty$. In the sequel, (i), (ii) and (iv) of Assumptions 1.4 are fulfilled for these $a(t)$, $b(t)$.

Theorem 1.4. *Let $t_0 > T_0$ and $x_0 \in (0,\ell)$. If Assumptions 1.4 are satisfied, then for the solution u of $NP(+)$ either $u(t,x_0) \equiv 0$ or there exists an infinite sequence $\{t_n\}$ such that $u(t_n,x_0) = 0$ and $u(r,x_0)u(s,x_0) < 0$ for every $r \in (t_{n-1},t_n)$ and $s \in (t_n,t_{n+1})$, where $\lim_{n\to\infty} t_n = \infty$.*

2. Globally oscillatory property

In this section we give a proof of Theorem 1.1. To this end, let us employ some weight function introduced in [2], which plays a role instead of the first positive eigenfunction to the operator $-A(t,x,\partial_x)$ in bounded domains (see [8]) and prepare a lemma to do so.

Put $|x| = \sqrt{\sum_{k=1}^n x_k^2}$ and $\psi(x) = (1+|x|^2)^{-\rho}$ for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $\rho > \frac{n}{2}$. Then ψ admits the following elementary properties got by easy calculations.

(i) $\psi \in H^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.
(ii) There is some constant $M > 0$ satisfying $|A(t,x,\partial_x)\psi(x)| \leqslant M\psi(x)$ in $I \times \mathbb{R}^n$.

Next, modify ψ by $\varphi(x) = \psi(\frac{\varepsilon}{M}x)$ for any constant $\varepsilon \in (0,M]$.

Proposition 2.1. *Let ε be any positive number. Then there exists a positive function $\varphi \in H^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that $A(t,x,\partial_x)\varphi(x) \leqslant \varepsilon\varphi(x)$ in $I \times \mathbb{R}^n$.*

The proof of Proposition 2.1 immediately follows from (i), (ii) and so we omit it.

Proof of Theorem 1.1. We may first assume that $J = [0,T]$ with $T > \frac{\pi}{\sqrt{b_0}}$. Let $\varepsilon = \frac{b_0}{2} - \frac{\pi^2}{2T^2} < M$ and $U(t) = \int_{\mathbb{R}^n} u(t,x)\varphi(x)\,dx$ in J. Then, by the equation $P_+u = 0$ and Green's formula

$$U''(t) = \int_{\mathbb{R}^n} \partial_t^2 u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^n} \bigl[Au - bu - f(u)\bigr]\varphi\,dx$$
$$= \int_{\mathbb{R}^n} \bigl[uA\varphi - bu\varphi - f(u)\varphi\bigr]\,dx.$$

Suppose that $u(t,x) \geqslant 0$ almost everywhere in $J \times \mathbb{R}^n$. Then, via $U(t) \geqslant 0$ and Proposition 2.1

$$\int_{\mathbb{R}^n} uA\varphi\,dx \leqslant \varepsilon U.$$

As well, from Assumptions 1.1

$$\int_{\mathbb{R}^n} (-bu\varphi)\,dx \leqslant -b_0U, \qquad \int_{\mathbb{R}^n} f(u)\varphi\,dx \geqslant 0.$$

So that

$$U''(t) \leqslant (\varepsilon - b_0)U(t)$$

in J, consequently

$$\int_0^T U''(t)\sin\omega t\,dt \leqslant (\varepsilon - b_0)\int_0^T U(t)\sin\omega t\,dt, \tag{2.1}$$

where $\omega = \sqrt{b_0 - 2\varepsilon}$ and recall that $\omega T = \pi$.

Meanwhile, thanks to integration by parts

$$\int_0^T U''(t)\sin\omega t\,dt = -\int_0^T U'(t)\omega\cos\omega t\,dt = -\omega^2\int_0^T U(t)\sin\omega t\,dt + \omega\bigl(U(T) + U(0)\bigr),$$

where it follows from $U(T)$, $U(0) \geqslant 0$ that

$$\int_0^T U''(t)\sin\omega t\,dt \geqslant (2\varepsilon - b_0)\int_0^T U(t)\sin\omega t\,dt. \tag{2.2}$$

Hence we have by (2.1) and (2.2)

$$\varepsilon\int_0^T U(t)\sin\omega t\,dt \leqslant 0,$$

further from $U(t)\sin\omega t \geqslant 0$ in J,

$$\int_0^T U(t)\sin\omega t\,dt = 0,$$

which implies $U(t)=0$ in J. However, this contradicts the non-triviality of $u(t,x)$ in $J\times\mathbb{R}^n$. Thus we have completed the proof. □

3. Non-oscillatory property

In this section we shall show Theorems 1.2 based on the method of characteristic curves. For any point $C(\hat{t},\hat{x})$ in $[t_0,T]\times\mathbb{R}$, we define the characteristic curves C_+, C_- of P by the solutions of the following ordinary differential equations

$$C_\pm:\quad \begin{cases} \dfrac{dx}{dt} = \pm\sqrt{a(t)}, \\ x(\hat{t}) = \hat{x}. \end{cases}$$

Let us denote $A=(t_0,\hat{x}-\int_{t_0}^{\hat{t}}\sqrt{a(s)}\,ds)$, $B=(t_0,\hat{x}+\int_{t_0}^{\hat{t}}\sqrt{a(s)}\,ds)$ and the dependence domain of C by

$$D = D(\hat{t},\hat{x}) = \left\{ (t,x)\in[t_0,T]\times\mathbb{R} \;\middle|\; |x-\hat{x}| \leqslant \int_t^{\hat{t}} \sqrt{a(s)}\,ds,\ t_0\leqslant t\leqslant \hat{t} \right\}.$$

Noting that $dt=0$ on the segment AB, $dx=-\sqrt{a(t)}\,dt$ on the curve BC and $dx=\sqrt{a(t)}\,dt$ on the curve CA, we have by Green's formula

$$\begin{aligned}
\int_D Pu\,dx\,dt &= \int_D \big(u_{tt}-a(t)u_{xx}\big)\,dx\,dt - \int_D F\big(u(t,x)\big)\,dx\,dt \\
&= \int_D \left(\frac{\partial}{\partial t}u_t - \frac{\partial}{\partial x}\big(a(t)u_x\big)\right) dx\,dt - \int_D F\big(u(t,x)\big)\,dx\,dt \\
&= -\int_{\partial D} \big(u_t\,dx + \big(a(t)u_x\big)\,dt\big) - \int_D F\big(u(t,x)\big)\,dx\,dt \\
&= -\int_D F\big(u(t,x)\big)\,dx\,dt - \int_{\mathrm{A}}^{\mathrm{B}} u_t\,dx - \int_{\mathrm{B}}^{\mathrm{C}} u_t\,dx + \big(a(t)u_x\big)\,dt - \int_{\mathrm{C}}^{\mathrm{A}} u_t\,dx + \big(a(t)u_x\big)\,dt \\
&= -\int_D F\big(u(t,x)\big)\,dx\,dt - \int_{\mathrm{A}}^{\mathrm{B}} u_t\,dx - \int_{\mathrm{B}}^{\mathrm{C}} u_t\big(-\sqrt{a(t)}\,dt\big) + \big(a(t)u_x\big)\frac{dx}{-\sqrt{a(t)}} - \int_{\mathrm{C}}^{\mathrm{A}} u_t\big(\sqrt{a(t)}\,dt\big) + \big(a(t)u_x\big)\frac{dx}{\sqrt{a(t)}} \\
&= -\int_D F\big(u(t,x)\big)\,dx\,dt - \int_{\mathrm{A}}^{\mathrm{B}} u_t\,dx + \int_{\mathrm{B}}^{\mathrm{C}} \sqrt{a(t)}u_t\,dt + \sqrt{a(t)}u_x\,dx - \int_{\mathrm{C}}^{\mathrm{A}} \sqrt{a(t)}u_t\,dt + \sqrt{a(t)}u_x\,dx \\
&= -\int_D F\big(u(t,x)\big)\,dx\,dt - \int_{\mathrm{A}}^{\mathrm{B}} u_t\,dx + \left(\sqrt{a(\mathrm{C})}u(\mathrm{C}) - \sqrt{a(\mathrm{B})}u(\mathrm{B}) - \int_{\mathrm{B}}^{\mathrm{C}} \frac{a'(t)}{2\sqrt{a(t)}}u\,dt\right) \\
&\quad - \left(\sqrt{a(\mathrm{A})}u(\mathrm{A}) - \sqrt{a(\mathrm{C})}u(\mathrm{C}) - \int_{\mathrm{C}}^{\mathrm{A}} \frac{a'(t)}{2\sqrt{a(t)}}u\,dt\right) \\
&= -\int_D F\big(u(t,x)\big)\,dx\,dt - \int_{\mathrm{A}}^{\mathrm{B}} u_t\,dx + 2\sqrt{a(\hat{t})}u(\mathrm{C}) - \sqrt{a(t_0)}u(\mathrm{A}) - \sqrt{a(t_0)}u(\mathrm{B}) - \int_{\mathrm{B}}^{\mathrm{C}} \frac{a'(t)}{2\sqrt{a(t)}}u\,dt - \int_{\mathrm{A}}^{\mathrm{C}} \frac{a'(t)}{2\sqrt{a(t)}}u\,dt.
\end{aligned}$$

Therefore, when $a(\hat{t})\neq 0$,

$$u(\hat{t},\hat{x}) = \frac{1}{2\sqrt{a(\hat{t})}}\left(\int_D F\big(u(t,x)\big)\,dx\,dt + \int_{\mathrm{A}}^{\mathrm{B}} u_t(x)\,dx + \sqrt{a(t_0)}\big(u(\mathrm{A})+u(\mathrm{B})\big) + \int_{\mathrm{B}}^{\mathrm{C}} \frac{a'(t)}{2\sqrt{a(t)}}u\,dt + \int_{\mathrm{A}}^{\mathrm{C}} \frac{a'(t)}{2\sqrt{a(t)}}u\,dt\right). \tag{3.1}$$

Now, suppose that u is not positive somewhere in $[0,T]\times\mathbb{R}$. Then there is a point $C(\hat{t},\hat{x})$ where $u(\hat{t},\hat{x})=0$ and $u(t,x)$ is positive for any point (t,x) with $t_0<t<\hat{t}$.

On the other hand, it follows from (3.1) that

$$0=\int\limits_D F\big(u(t,x)\big)\,dx\,dt+\int\limits_A^B u_1(x)\,dx+\sqrt{a(t_0)}\big(u_0(\mathrm{A})+u_0(\mathrm{B})\big)+\int\limits_B^C \frac{a'(t)}{2\sqrt{a(t)}}u\,dt+\int\limits_A^C \frac{a'(t)}{2\sqrt{a(t)}}u\,dt. \tag{3.2}$$

But the right-hand side of (3.2) is positive from Assumptions 1.2. So we get a contradiction. The proof has just finished. □

4. Pointwise oscillatory property

Proof of Theorem 1.3. For the proof of Theorem 1.3, according to [15] and [2], the equation $P_+u=0$ is transformed into an inhomogeneous wave equation in one space dimension.

We begin with the change of the strongly elliptic operator $a(t)\Delta_x$ to the Laplacian Δ_x in $[t_0,T)$. To this end, define a function s of t in $[t_0,T)$ by

$$s=\int\limits_{t_0}^{t}\sqrt{a(\tau)}\,d\tau.$$

By $a'(t)=\sqrt{a(t)}a(t)_s$, if we put $w(s,x)=u(t,x)$, then w solves the semilinear wave equation

$$w_{ss}-\Delta_x w+p(s)w_s+q(s)w+g(w)=0$$

in $[0,S)\times B_\ell$, where $p(s)=\frac{(\sqrt{a(t)})_s}{\sqrt{a(t)}}$, $q(s)=\frac{b}{a}$, $g(w)=\frac{f(w)}{a}$ and $S=\int_{t_0}^T\sqrt{a(t)}\,dt$. Next, to extract the 1st order term $p(s)w_s$, taking advantage of Liouville's transformation

$$y(s,x)=w(s,x)\exp\Bigg(\frac{1}{2}\int\limits_0^s p(t)\,dt\Bigg),$$

we have the semilinear wave equation

$$y_{ss}-\Delta_x y+Q(s)y+R(y)=0, \tag{4.1}$$

where $Q(s)=q(s)-\frac{p'(s)}{2}-\frac{p(s)^2}{4}$ and

$$R(y)=g\Bigg(y\exp\Bigg(-\frac{1}{2}\int\limits_0^s p(t)\,dt\Bigg)\Bigg)\exp\Bigg(\frac{1}{2}\int\limits_0^s p(t)\,dt\Bigg).$$

In our case, since $p=\frac{a_t}{2a\sqrt{a}}$ and $p'(s)=\frac{(\sqrt{a})_{ss}}{\sqrt{a}}-\frac{(\sqrt{a})_s^2}{a}=\frac{a_{tt}}{2a^2}-\frac{3a_t^2}{4a^3}$, we find

$$Q(s)=\frac{b}{a}-\frac{a_{tt}}{4a^2}+\frac{5a_t^2}{16a^3}, \tag{4.2}$$

where note that $(\sqrt{a})_s=\frac{a_t}{2a}$ and $(\sqrt{a})_{ss}=\frac{a_{tt}}{2a\sqrt{a}}-\frac{a_t^2}{2a^2\sqrt{a}}$. So, if we take the function $h(s,r)$ as $h(s,r)=v(\varphi(s),r)\exp(\frac{1}{2}\int_0^s p(t)\,dt)$, then Eq. (4.1) is rewritten like

$$h_{ss}-h_{rr}-\frac{2}{r}h_r+Q(s)h+R(h)=0 \tag{4.3}$$

in $[0,S)\times(0,\ell)$, where v is a radially symmetric solution to $P_+u=0$ and φ is the inverse function of $t\mapsto s$. Moreover, by introducing $z(s,r)=rh(s,r)$, similarly to Lemma 2.2 in [2], (4.3) turns out the semilinear wave equation in one space dimension

$$z_{ss}-z_{rr}+Q(s)z+rR(z/r)=0. \tag{4.4}$$

Then the resulting initial condition is also changed to

$$\begin{cases} z(0,r)=rv(t_0,r),\\ z_s(0,r)=\dfrac{(\sqrt{a(0)})_t r}{2a(0)}v(t_0,r)+\dfrac{r}{\sqrt{a(0)}}v_t(t_0,r)\end{cases}$$

in $[0,\ell]$, but the homogeneous Dirichlet boundary condition is unchanged, namely

$$z(s,r)=0$$

on $[0, S) \times \{0, \ell\}$. Hence, taking account that $Q(s) \geqslant 0$, $rR(z/r)$ is odd and nondecreasing in z from (4.2) and Assumptions 1.3, we can apply Theorem 2.2.1 in [1] or Theorem 3.1 in [14] to (4.4) and have obtained the conclusion. □

Remark 4.1. In general space dimensions n, if we set $z(s,r) = r^{(n-1)/2}h(s,r)$, then (4.4) changes to the equation

$$z_{ss} - z_{rr} + \left[Q(s) + \frac{(n-1)(n-3)}{4r^2}\right]z + r^{(n-1)/2}R(z/r) = 0.$$

However, since $\frac{1}{r^2} \notin L^{\infty}_{\mathrm{loc}}(\mathbb{R})$, we cannot adopt any result in [1] and [14] except $n = 1$ and 3. This is the reason why the space dimensions is only restricted to $n = 3$.

Now, let us consider the linear ordinary differential equation of second order in $[t_0, \infty)$

$$w''(t) + c(t)w(t) = 0. \tag{4.5}$$

We know many results on the distribution of zeros of solutions to (4.5) (see [10–12]). As one of them we cite the following fact for a proof of Theorem 1.4.

Lemma 4.1. *(See (I) in §3 of [12].) Let $c(t)$ a real-valued, piecewise continuous function defined on $[t_0, \infty)$. If for any fixed $T_1 \geqslant t_0$,* (iii) *of Assumptions* 1.4 *is satisfied for $t \in [T_1, \infty)$, then every nontrivial solution of* (4.5) *has countably infinite number of zeros in $[T_1, \infty)$.*

Proof of Theorem 1.4. The proof is performed by the almost same fashion as in that of Theorem 1.4 in [14], therefore we shall sketch the main steps and formulae.

At first, by the same reduction as in the proof of Theorem 1.3, the equation $P_+u = 0$ is transformed into the semilinear wave equation $y_{ss} - y_{xx} + Q(s)y + R(y) = 0$. Then, remark that the initial condition is also changed, but the homogeneous Neumann condition is unchanged, namely $y_x(s, 0) = y_x(s, \ell) = 0$ on $[0, \infty)$. So, let $y : [0, \infty) \times [0, \ell] \to \mathbb{R}$ be a solution of this mixed problem. And define $Y(s,x)$ by the extension of $y(s,x)$ as even function with respect to x so that $Y(s,x) = y(s, -x)$ for $x \in [-\ell, 0]$ and $Y(s, x+2\ell) = Y(s,x)$ for $(s,x) \in [0,\infty) \times \mathbb{R}$. Then Y is even in x and $Y_{ss} - Y_{xx} + Q(s)Y + R(Y) = 0$. Next, if we put $v(s,x) = Y(s,x) + Y(s, 2x_0 - x)$, then

$$v_x(s, x_0) = 0. \tag{4.6}$$

Moreover, set

$$h(s,x) = \begin{cases} \frac{R(Y(s,x)) + R(Y(s,2x_0-x))}{v(s,x)} & \text{if } v(s,x) \neq 0, \\ 0 & \text{if } v(s,x) = 0. \end{cases}$$

Then $h(s,x) \geqslant 0$ in $[0,\infty) \times \mathbb{R}$ and $h(s,x) \in L^{\infty}(J \times \mathbb{R})$ for every finite interval J in $[0, \infty)$ from (v) and (vi) of Assumptions 1.4. As well, v solves the linearized wave equation

$$v_{ss} - v_{xx} + Q(s)v + h(s,x)v = 0 \tag{4.7}$$

in $[0,\infty) \times \mathbb{R}$.

Now we suppose that for any fixed s_1 and s_2 with $0 \leqslant s_1 < s_2 \leqslant \infty$

$$y(s, x_0) \geqslant 0$$

holds for all $s \in [s_1, s_2)$. Then we shall verify that $s_2 < \infty$ unless $u(t, x_0)$ identically vanishes for all $t \in [t_0, \infty)$. After this assertion, $u(t_2, x_0) = 0$ and $u(t, x_0) < 0$ for all $t > t_2$ in a small neighborhood of $t_2 = \varphi(s_2)$.

So, from now suppose that $s_2 = \infty$, namely $y(s, x_0) \geqslant 0$ for all $t \in [s_1, \infty)$. At first, by the definition of Y, since $Y(t, x_0) \geqslant 0$ in $[s_1, \infty)$,

$$v(s, x_0) = 2Y(s, x_0) \geqslant 0 \tag{4.8}$$

in $[s_1, \infty)$. Hence, because of (4.6) and (4.8), we may apply Lemma 2.1 in [14] to (4.7) and eventually $v(s,x) \geqslant 0$ in $[0,\infty) \times \mathbb{R}$. In particular, introducing $V(s) = \int_{-\ell}^{\ell} v(s,x)\,dx$, we have $V(s) \geqslant 0$ in $[0, \infty)$. Besides, in account of the homogeneous Neumann boundary condition for y and the 2ℓ-periodicity of v in x,

$$\int_{-\ell}^{\ell} v_{xx}(s,x)\,dx = 0.$$

Therefore, from Eq. (4.7)

$$V''(s) = -Q(s)V(s) - \int_{-\ell}^{\ell} h(s,x)v(s,x)\,dx,$$

and thanks to $\int_{-\ell}^{\ell} h(s,x)v(s,x)\,dx \geqslant 0$, we obtain the ordinary differential inequality

$$V''(s) + Q(s)V(s) \leqslant 0 \tag{4.9}$$

in $[0,\infty)$. Also, define W by the solution to the Cauchy problem

$$\begin{cases} W''(s) + Q(s)W(s) = 0 \quad \text{in } [0,\infty), \\ W(0) = V(0), \quad W'(0) = V'(0). \end{cases} \tag{4.10}$$

We recall that W has countably infinite number of zeros in $[s_3,\infty)$ for any fixed $s_3 \in [0,\infty)$ by (iv) of Assumptions 1.4. As is well known, it follows from (4.9) and (4.10) that

$$0 \leqslant V(s) \leqslant W(s)$$

at each $s \geqslant 0$ when $W(s) \geqslant 0$. Consequently, since there exist zeros of W in $[s_1,\infty)$, $V(s_3) = 0$ and $v(s_3,x) = 0$, in particular $v_x(s_3,x) = 0$ for some $s_3 \in [s_1,\infty)$ and all $x \in \mathbb{R}$. Meanwhile, because $v(s,x) \geqslant 0$ for all $x \in \mathbb{R}$, for any fixed $x \in \mathbb{R}$, $v(s_3,x) = 0$ is a minimum of the function $s \mapsto v(s,x)$. So we see that $v_s(s_3,x) = 0$.

Now, introducing the energy function $E(s)$ for v by

$$E(s) = \int_0^{2\ell} \left[\left|v_s(s,x)\right|^2 + Q(s)\left|v_x(s,x)\right|^2 + \left|v(s,x)\right|^2 \right] dx,$$

due to (4.7) and Schwarz' inequality we know the resulting energy estimate

$$E(s) \leqslant K(s)E(s_3),$$

where $K(s) \geqslant 1$ is some increasing continuous function. Therefore, from $v(s_3,x) = v_s(s_3,x) = v_x(s_3,x) = 0$, $v(s,x) = 0$ in $[s_1,\infty) \times \mathbb{R}$. In particular, for $x = x_0$, we get $u(t,x_0) = 0$ in $[\varphi(s_1),\infty)$.

Similarly, if $u(t,x_0) \leqslant 0$ is valid in $[t_1,\infty)$, then we can conclude that $u(t,x_0) = 0$ in $[t_1,\infty)$. Thus the proof has just finished. □

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